

Lightsheets, Branes and the Covariant Entropy Bound

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Abstract

We investigate the geometry of lightsheets comprised of null geodesics near a brane. Null geodesics which begin parallel to a brane a distance d away are typically gravitationally bound to the brane, so that the maximum distance from the geodesic to the brane never exceeds d . The geometry of resulting lightsheets is similar to that of the brane if one coarse grains over distances of order d . We discuss the implications for the covariant entropy bound applied to brane worlds.

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It is possible that the maximum information content of a spacetime region is related to its surface area [1, 2, 3]. The idea has its origins in the proposal of Bekenstein that the area of a black hole is proportional to its entropy [4], and that black holes obey a generalized second law of thermodynamics (GSL) [5]. A covariant generalization of these ideas [6, 7] has passed a number of theoretical tests, and suggests a deep relationship between geometry and information which arises due to quantum gravity. This covariant entropy bound (henceforth, the covariant bound) can be stated as follows:

Let $A(B)$ be the area of an arbitrary $D - 2$ dimensional spatial surface B , which need not be closed. A $D - 1$ dimensional hypersurface $L(B)$ is the light-sheet of B if $L(B)$ is generated by light rays extending orthogonally from B , which have non-positive expansion everywhere on $L(B)$. Let $S(L)$ be the total entropy of matter which intersects $L(B)$. Then $S(L) \leq \frac{1}{4}A(B)$.

For simple cases, such as a suitable closed spacelike surface surrounding a weakly gravitating system, the covariant bound reduces to the usual area bound.

In [8], the covariant bound was shown to be violated in brane world scenarios [9, 10] in which the fundamental scale of quantum gravity is $M_* \sim \text{TeV}$. Consider a spacelike region V of extent r on the 3-brane and thickness l in the orthogonal extra dimensions (see Fig. 1). The boundary of V consists of components whose surface areas scale as $r^3 l^{(D-5)}$ and $r^2 l^{(D-4)}$. The first surface component is obtained by setting the extra-dimensional coordinates at their extreme (boundary) values and allowing the coordinates $\{x_{1-3}\}$ to vary throughout the intersection of V with the 3-brane. (This is shown as the shaded region in Fig. 1.) The second is obtained by setting $\{x_{1-3}\}$ at their extreme values (i.e., the boundary on the 3-brane) and letting the extra-dimensional coordinates to vary over a range of size l . (This is indicated by the unshaded, but lined, region in the figure.)

The surface B in the covariant bound is taken as the *second* part of the boundary of V , the one whose area scales as $r^2 l^{(D-4)}$ (note that B need not be closed). In Fig. 1 this appears as the unshaded portion of the cylindrical surface. Let V have the same shape as the brane, with thickness l of order M_*^{-1} (the minimum thickness possible; the same as that of the brane), so that its surface area is of order r^2 in M_* units. The light sheet $L(B)$ is comprised of null geodesics emanating orthogonally from B . These geodesics intersect all of the ordinary matter in V , so the entropy $S(L)$ is simply that of the ordinary matter in V . In [8] it was shown that $S(L)$ can exceed $A(B)$ in M_* units for systems such as a supernova core or the early universe.

Taking B to be the same thickness of the brane avoids the question of whether the gravitational pull of the brane in the extra dimension focuses the rays of $L(B)$ to a caustic

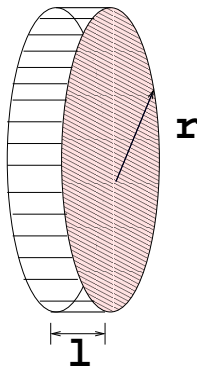


Figure 1: $D - 2$ dimensional surface B (the unshaded region of thickness l) in the brane scenario.

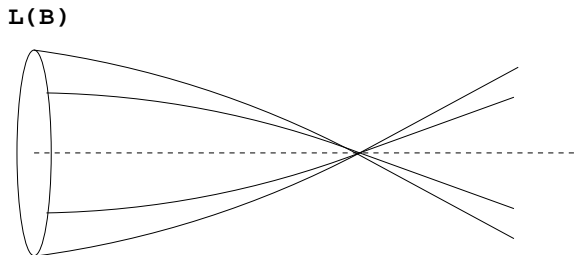


Figure 2: Null geodesics focused into caustic.

before they reach the center of the fiducial volume. The condition of non-expansion used in defining $L(B)$ would cause it to terminate at a focal point, and much of the matter would never intersect $L(B)$. (See Fig. 2) Because $l_* \sim M_*^{-1}$ is the fundamental length scale of quantum gravity and also the thickness of the brane, we do not consider any focusing of $L(B)$. (There is likely no meaning to distances less than l_* [11].) However, it remains to be seen whether this $L(B)$ can be obtained as a smooth limit of lightsheets $L(B')$ resulting from surfaces B' with larger extent in the extra dimensions. Otherwise, one might consider the construction used in [8] to be a degenerate limit¹. We address this issue below, and show that a lightsheet $L(B)$ with the same geometry as the brane can be obtained as the smooth limit of a family of lightsheets $L(B')$, if a slightly modified (coarse grained) definition of lightsheets is adopted. The gravitational binding of null geodesics to a nearby brane is a key component of this analysis.

In RS geometry [10] the metric is given as [12]

$$ds^2 = dy^2 + e^{-2|y|/R} \eta_{\mu\nu} dx^\mu dx^\nu. \quad (1)$$

The equations for geodesics emanating from orthogonally from B , and parallel to the TeV

¹S. Hsu thanks S. Giddings for emphasizing this point.

brane located at the origin, $y = 0$, are given as, for $y \geq 0$,

$$\frac{d^2 y}{ds^2} + \frac{2}{R} e^{-2y/R} \left[\left(\frac{dt}{ds} \right)^2 - \left(\frac{d\vec{x}}{ds} \right)^2 \right] = 0 \quad (2)$$

$$\frac{d^2 t}{ds^2} + \frac{dt}{ds} \frac{dy}{ds} = 0 \quad (3)$$

$$\frac{d^2 \vec{x}}{ds^2} - \frac{d\vec{x}}{ds} \frac{dy}{ds} = 0 . \quad (4)$$

For this special geometry, solutions with $dy/ds = 0$ always exist if the velocity vector $u^a = (dt/ds, d\vec{x}/dt)$ is null (i.e., the difference in brackets in equation (2) vanishes). Therefore, it is possible to have light rays which travel parallel to the brane without being focused. If the rays remain parallel to the brane, the lightsheet $L(B)$ is easily seen to result from the smooth limit of lightsheets $L(B')$, which are similar to $L(B)$ but with larger extent in the extra dimensions.

In general the metric on the brane is not simply $\eta_{\mu\nu}$, i.e., if there is matter or energy density on the brane. In the examples considered in [8], the metric on the brane is either the Robertson-Walker metric of the early universe, or that of a supernova interior, and may have t or \vec{x} dependence. In such cases we expect that geodesics may be bent toward the brane by gravitational attraction. Consider a small perturbation to the metric (1), and suppose that it forces the solution to the geodesic equations to have non-zero $dy/ds \neq 0$. Since the deviation from (1) is assumed to be a perturbation, we deduce the leading order behavior as follows. The general solution to (2-4) for arbitrary initial conditions satisfies

$$\frac{dt}{ds} = C_0 e^{-y} \quad (5)$$

$$\frac{dx_i}{ds} = C_i e^y \quad (6)$$

$$\frac{1}{2} \left(\frac{dy}{ds} \right)^2 = E - V(y) , \quad (7)$$

where C_0, C_i and E are constants of integration, and

$$V(y) = \frac{-2C_0^2}{2R+2} e^{-(2/R+2)y} - \frac{2C_i^2}{2R-2} e^{(2-2/R)y} . \quad (8)$$

We see that in general the geodesics hit the origin $y = 0$ exponentially quickly.

However, if a light ray, starting parallel to a brane M , hits M after some time, it will be forever bound to M . The proof is as follows (see Fig. 3). Let M be infinite and uniform. Assume the D -dimensional universe is Z_2 symmetric under reflections through M . Let a light ray start parallel to M at height d and intersect M from above at point P with angle α . Consider the time reversed trajectory: it shows that a ray leaving M with angle α will

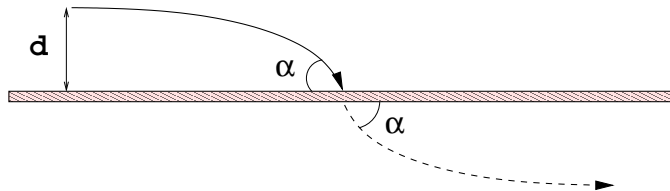


Figure 3: Focusing of light rays due to the gravitational force of the brane.

eventually end up parallel to M at height d . Now, when the original ray crosses through M from above at point P it then leaves M from below at angle α . By symmetry (using Z_2 reflections through M itself, and through a hyperplane orthogonal to M which passes through P), this ray will end up parallel to the brane at a height $-d$. This argument, when repeated, implies the ray is bound forever to M and never more than distance d away from M . Although we used translation invariance in the above argument (to justify the Z_2 reflection through the hyperplane orthogonal to M at P), we expect corrections due to small deviations from translation invariance to be small. Indeed, the question of whether a light ray is bound to the brane (i.e., whether it can escape to infinity) is ultimately an energetic one, and hence not sensitive to small rearrangements of the energy on the brane.

The binding of null geodesics to a brane seems outside the range of behaviors imagined for lightsheets in the formulation of the covariant bound [3]. In the original formulation, an important objective was to give a criteria for terminating a lightsheet, since an infinite lightsheet might intersect an infinite amount of entropy and render the bound problematic. It seems to have been implicitly assumed that focusing of light rays, or the formation of a caustic, would inevitably lead to subsequent expansion and divergence of the lightsheet area (as depicted in Fig. 2). The argument above shows that focusing does not necessarily imply divergent expansion.

Given the semiclassical spirit of the covariant entropy bound, it seems reasonable to use a *coarse grained* definition of lightsheets $L(B)$. In particular, in deciding where to terminate a lightsheet, the expansion $\theta(\lambda)$

$$\theta(\lambda) \equiv \frac{d\mathcal{A}/d\lambda}{\mathcal{A}} \quad (9)$$

can be allowed to be slightly positive for a short interval, as long as the coarse grained $\theta(\lambda)$ is not positive. (Alternatively, $\theta(\lambda)$ could be allowed to become slightly positive, but with magnitude smaller than a coarse graining scale.) This definition does *not* admit lightsheets with diverging area (as displayed in Fig. 2), and it reduces to the usual covariant bound in simple cases. However, using this coarse grained definition, the light sheets bound to the brane do not terminate due to focusing. If a coarse graining scale of order d is adopted, they produce a uniform sheet of thickness $2d$ (See Fig. 4.) According to any coarse grained

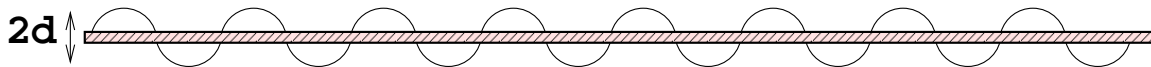


Figure 4: Light rays bound to the brane.

definition, the $L(B)$ in the brane world construction of [8] (which has the same geometry as the brane itself) is the smooth limit of a family of lightsheets $L(B')$ resulting from surfaces B' with decreasing thickness in the extra dimension.

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